Balanced Incomplete Block Design with Control Treatment

Sawsan Hassab Elrasoul Babiker Mohamed

Dep. of Mathematics & Physics , Faculty of Education , University of Gezira , e-mail:Sawsanbabiker@gmail.com

ABSTRACT

The theory of design and analysis of experiments has been primarily developed by statisticians engaged mostly in agricultural research. The theory has now found applications in other fields of research, because it is based on general principles concerning the statistical behavior of observations which arise either freely in nature or in artificial laboratory conditions (Badrldin and Kshirsagar, 1990).Incomplete block designs were developed to suit experiments where the number of experimental units per block is less than the number of treatments. The concept of incomplete block designs augmented by a control was first introduced by Das (1954). He considered the case where ($\theta \geq 1$) new treatments are introduced into the design and all included in each block. He concluded that θ should be chosen as low as possible to keep the block size to a reasonable level. Pearce (1960) considered the case where a *Balanced Incomplete Block* (B. I. B.) design is supplemented by a control, where the test treatments are replicated r times except the control which is replicated r_0 times, while all pairs of treatments occurs λ times in blocks, the supplemented treatment occurs λ_0 times with any other treatment(Ture,1982). Pesek (1974) considered the case of a B. I. B. design having an extra control in each block. He utilized Rao's general formulas to obtain the variances of the elementary treatment contrast between any pair of test treatment and any treatment and the control. He also obtained the efficiency factor of this design and showed that this design is more efficient than a B. I. B. design for comparing treatments with a control, but is less efficient for pair wise comparisons between the test treatments. Many times, for important new drugs and for serious diseases, investigations are carried out simultaneously at various locations under different climatic conditions, for a series of treatments. In such cases, the result of all such investigations need to be combined to produce an overall estimate of the effect of treatment contrast to the control.

In this study the case where a control is added to each block in a B. I. B. design is considered. The variances of the elementary treatment contrast between any pair of test treatments and any treatment and the control were obtained. The general theory of inter & intra -block estimates of treatment effect when a B. I. B. design is augmented by a control was investigated, the general theory of a balanced incomplete block design is given where a control is added to each block. The estimation of weight for combining inter and intra-block estimates is also dealt with, and the weight for combining inter-block estimates of treatment contrast was also considered. The object of this study is to get a minimum variance of the treatment contrast between the inter and intra block estimates when a control treatment is added to each block in B. I. B. design.

Key words: Control, B. I. B. Design, inter-block, intra- block, treatment contrast

INTRODUCTION

Comparison between treatments and the control are of frequent interest in many fields of research because the control treatment plays a pivotal role, that every test treatment is compared to it. It is natural for the researchers to believe that more observations should be allocated to the control treatment than to each test treatments. Pearce (1960) suggested that the control should be more frequently replicated than the other test treatments by a ratio equal to the square root of the number of test treatments. Dunnett (1964) has given examples from the biological and pharmacological fields where new designs for this kind of situation are needed when the control is to be used in every block. The usual balanced incomplete block design [B.I.B.D.] is not appropriate and needs to be modified. Bechhofer and Tamhane (1981) have provided designs of this type where the test treatments are replicated r times except the control which is replicated r_0 times while all pairs of treatments occur λ times in blocks, the control treatment occurs λ_0 times with any other test treatment. None parametric methods for determining differences in treatment effects have been proposed for (BIBD) by Durbin (1951), which is useful for testing treatment effects versus control.

Notation

Consider a balanced incomplete block design (B.I.B.D) with an extra control in each block. The v + 1 treatments are arranged in b blocks such that each block contains k + 1 experimental units. Let the treatments be denoted by, $t_0, t_1, \ldots, t_v$ with t_0 being the control treatment and the others are test treatments. The test treatments are replicated r times and the control is replicated b times.

Let N = [n_{ij}] to be the treatment-block incidence matrix having v + 1 rows and b columns where: n_{ij} the number of times the i-th treatment occurs in the j-th block $[i = 0,1,2,\ldots\ldots v;\ j = 1,2,3,\ldots\ldots b]$ Then, the treatments-block incidence matrix is

$$NN' = \begin{bmatrix} b & rE_{1v} \\ rE_{v1} & (r-\lambda)I_v + \lambda E_{vv} \end{bmatrix}$$

It will be assumed that:

$$\sum_{j=1}^{b} n_{0j} = r_0 = b \qquad \sum_{j=1}^{b} n_{ij} = r;\ i = 1,2,\ldots\ldots,v \qquad (2.1)$$

$$\sum_{j=1}^{b} n_{0j}^{\ 2} = b\ ; \qquad \sum_{j=1}^{b} n_{ij}^{\ 2} = r\ ;\ i = 1,2,\ldots\ldots,v \qquad (2.2)$$

$$\sum_{j=1}^{b} n_{0j}\, n_{ij} = r\ ;\ \sum_{j=1}^{b} n_{ij}\, n_{uj} = \lambda \qquad (2.3)$$

$$\sum_{i=0}^{v} n_{ij} = k+1 \;; \quad j = 1,2,........,b \tag{2.4}$$

Notice that: $NN' E_{v+1,1} = (k+1)NE_{b1} = (k+1)\underline{r}$ (2.5)

Where $\underline{r}$ is column of vector of elements? $r_0, r_1, ..., r_v$

The Model

For the balanced incomplete block design considered in section (2),
The intra-block model is:

$$Y_{ij} = t_i + \beta_j + e_{ij} \tag{3.1}$$

Where:

Y_{ij} Is the yield of the i-th treatment in the j-th block, (if $n_{ij} = 1$)

t_i fixed treatment effect?

B_j Fixed block effect.

e_{ij} is the usual error which is assumed to be normally distributed with zero mean and variance σ^2, the errors are assumed to be independent $e_{ij} \sim NI(0, \sigma^2)$. According to Cochran and Cox, (1957), the model assumes no interaction between blocks and treatments. The intra-block normal equation are derived by minimizing:

$$Q = \sum_{i}^{v} \sum_{j}^{b} (Y_{ij} - t_i - \beta_j)^2 \tag{3.2}$$

With respect to t_i and B_j $(i = 0,1,2,......,v \quad ; \quad j = 1,2,....,b)$.

$$\frac{\partial Q}{\partial t} = -2\sum_{j=1}^{b} (Y_{ij} - \hat{t}_i - \hat{\beta}_j) = 0$$

$$\sum_{j=1}^{b} Y_{ij} = \sum_{j=1}^{b} n_{ij}\hat{t}_i + \sum_{j=1}^{b} n_{ij}\hat{\beta}_j \tag{3.3}$$

$$Y_{i.} = T_i = r\hat{t}_i + \sum_{j=1}^{b} n_{ij} \hat{\beta}_j$$

$T_{i,}$ The sum of all observations on the i-th treatment,

$$\frac{\partial Q}{\partial \beta} = -2\sum_{i=0}^{v}(Y_{ij} - \hat{t}_i - \hat{\beta}_j) = 0$$

$$\sum_{i=0}^{v} Y_{ij} = \sum_{i=0}^{v} n_{ij}\hat{t}_i + \sum_{i=0}^{v} n_{ij}\hat{\beta}_j \qquad (3.4)$$

$$B_j = Y_{.j} = \sum_{i=0}^{v} n_{ij}\ \hat{t}_i + (k+1)\hat{\beta}_j$$

B_j The sum of all observations on the j-th block, from (3.3),

$$\hat{\beta}_j = \frac{1}{k+1}(Y_{.j} - \sum_{i=0}^{v} n_{ij}\ \hat{t}_i)$$

Substitute this in (3.3) we get:

$$Y_{i.} = r\hat{t}_i + \sum_{j=1}^{b} n_{ij}\left(\frac{1}{k+1}(Y_{.j} - \sum_{i=1}^{v} n_{ij}\ \hat{t}_i)\right)$$

$$Y_{i.} - \frac{1}{k+1}\sum_{j=1}^{b} n_{ij} Y_{.j} = r\hat{t}_i - \frac{1}{k+1}\sum_{j=1}^{b} n_{ij}\sum_{i=0}^{v} n_{ij}\hat{t}_i$$

$$Q_i = T_i - \frac{1}{k+1}\sum_{j=1}^{b} n_{ij} B_j = \hat{r}t_i - \frac{1}{k+1}\sum_{j=1}^{b} n_{ij}\sum_{i=0}^{v} n_{ij}\hat{t}_i$$

$$= \left(\frac{rk+\lambda}{k+1}\right)\hat{t}_i - \frac{r}{k+1}\hat{t}_0 - \frac{\lambda}{k+1}\sum \hat{t}_u \qquad (3.5)$$

$$Q_0 = (b - \frac{b}{k+1})\hat{t}_0 - \frac{r}{k+1}\sum_{u \neq control} \hat{t}_u \qquad (3.6)$$

$$= \frac{bk}{k+1}\hat{t}_0 - \frac{r}{k+1}\sum_{u \neq contro;} \hat{t}_u$$

Using the constrain $\sum_{u \neq contral}^{v} t_u = 0$ (3.7)

Therefore:

$$\hat{t}_o = \left(\frac{k+1}{bk}\right) Q_o \qquad (3.8)$$

$$Q_i = \left(\frac{rk+\lambda}{k+1}\right)\hat{t}_i - \frac{r}{k+1}\hat{t}_o \quad ; \quad rk+\lambda = \lambda v + r \qquad (3.9)$$

$$= \left(\frac{\lambda v + r}{k+1}\right)\hat{t}_i - \frac{1}{v}(Q_o)$$

$$Q_i + \frac{1}{v}(Q_o) = \left(\frac{\lambda v + r}{k+1}\right)\hat{t}_i$$

$$\hat{t}_i = \left[Q_i + \frac{1}{v}(Q_o)\right]\left(\frac{k+1}{\lambda v + r}\right) \qquad (3.10)$$

Therefore

$$\hat{t}_i - \hat{t}_o = \left(\frac{k+1}{\lambda v + r}\right)\left[Q_i + \frac{1}{v}(Q_o)\right] - \left(\frac{k+1}{bk}\right)Q_o$$
$$= \left(\frac{k+1}{\lambda v + r}\right)Q_i + \left(\frac{k+1}{v(\lambda v + r)}\right)Q_o - \left(\frac{k+1}{bk}\right)Q_o$$
$$= \left(\frac{k+1}{\lambda v + r}\right)Q_i + \left[\frac{k+1}{v(\lambda v + r)}\right]Q_o - \left(\frac{k+1}{vr}\right)Q_o$$
$$= \left(\frac{k+1}{\lambda v + r}\right)Q_i + \left[\frac{(k+1)(r - \lambda v - r)}{vr(\lambda v + r)}\right]Q_o$$
$$= \left(\frac{k+1}{\lambda v + r}\right)\left[Q_i - \frac{\lambda}{r}Q_o\right] \qquad (3.11)$$

The intra-block normal equations are thus:

$$\underline{Q} = C\underline{\hat{t}} \qquad (3.11)$$

$$\underline{Q} = \underline{T} - \frac{1}{k+1} N \underline{B}$$

Where:

$\underline{T} = (v+1)\times 1$ Vector of treatment totals (3.12)

$\underline{B} = b\times 1$ Vector of block totals (3.13)

$\underline{Q} = (v+1)\times 1$ vector of adjusted treatment totals (3.14)

$$C = diag(r_0, r_1, r_2 \ldots\ldots\ldots r_v) - \frac{1}{k+1} NN' \qquad (3.15)$$

$$C = \frac{1}{k+1}\begin{bmatrix} bk & -rE_{1v} \\ -rE_{v1} & (r+\lambda v - \lambda)I_v - \lambda J_v \end{bmatrix} \qquad (3.16)$$

And $\underline{Q} = C\underline{\hat{t}}$, then

$$\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ . \\ . \\ Q_v \end{bmatrix} = \begin{bmatrix} b & 0 & 0 & \ldots. & .. & 0 \\ 0 & r_1 & \ldots. & .. & .. & 0 \\ 0 & .. & r_2 & .. & \ldots & .. \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & \ldots & .. & 0 \\ 0 & \ldots & .. & \ldots & 0 & r_v \end{bmatrix} - \frac{1}{k+1}\begin{bmatrix} b & r & r & r & \ldots & r \\ r & rk & \lambda & \lambda\ldots & \ldots & \lambda \\ r & \lambda & rk & \lambda\ldots & \ldots & \lambda \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ r & \lambda & \lambda\ldots & \lambda & rk & \lambda \\ r & \lambda & \lambda\ldots & \ldots. & \lambda & rk \end{bmatrix}$$

(3.17)

$$\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ . \\ . \\ Q_v \end{bmatrix} = \frac{1}{k+1}\begin{bmatrix} bk & -r & -r & -r\ldots & \ldots & -r \\ -r & rk & -\lambda & -\lambda\ldots & \ldots & -\lambda \\ -r & -\lambda & rk & -\lambda\ldots & \ldots & -\lambda \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -r & -\lambda & -\lambda\ldots & -\lambda & rk & -\lambda \\ -r & -\lambda & -\lambda\ldots & \ldots. & -\lambda & rk \end{bmatrix}\begin{bmatrix} t_0 \\ t_1 \\ t_2 \\ . \\ . \\ t_v \end{bmatrix}$$

$$Q_0 = \frac{1}{k+1}\left[(bk)\hat{t}_0 + (-r)\sum_{i=1}^{v}\hat{t}_i\right]; \qquad \sum_{i=1}^{v}\hat{t}_i = 0$$

$$Q_0 = \left(\frac{bk}{k+1}\right)\hat{t}_0$$

Therefore: $\hat{t}_0 = \left(\frac{k+1}{bk}\right) Q_0$ (3.18)

$$Q_i = \frac{1}{k+1}\left[(-r)\hat{t}_0 + (rk)\hat{t}_i - \lambda\sum_{v}^{v}\hat{t}_v + \lambda\hat{t}_i\right]$$

$$Q_i = \left(\frac{\lambda v + r}{k+1}\right)\hat{t}_i - \frac{1}{v}(Q_0)$$

Therefore

$$\hat{t}_i = \left[Q_i + \frac{1}{v}(Q_0)\right]\left(\frac{k+1}{\lambda v + r}\right) \qquad (3.19)$$

$$\hat{t} = C$$

Note that:

$$E(\hat{t}_0) = E\left(\frac{k+1}{bk}Q_0\right) = t_0 - \bar{t}, \qquad where: \bar{t} = \frac{1}{v}\sum_{u \neq control} t_u$$

$$E(\hat{t}_i) = \frac{k+1}{rk+\lambda}\left(E(Q_i) + \frac{1}{v}E(Q_0)\right) = \hat{t}_i - \bar{t} \qquad (3.20)$$

$$\text{var}(\hat{t}_0) = \text{var}\left(\frac{k+1}{bk}Q_0\right) \qquad (3.21)$$

$$= \left(\frac{k+1}{bk}\right)\sigma^2$$

$$\text{var}(\hat{t}_i) = \text{var}\left(\frac{k+1}{rk+\lambda}\right)\left(Q_i + \frac{1}{v}Q_0\right)$$

$$= \left(\frac{k+1}{rk+\lambda}\right)^2\left[\text{var}(Q_i) + \frac{1}{v^2}\text{var}(Q_0) + 2\,\text{cov}(Q_i, \frac{1}{v}Q_0)\right],$$

$$\text{var}(Q_i) = \text{var}\left[T_i - \frac{1}{k+1}\sum_{j}^{b} n_{ij}B_j\right]$$

$$= \text{var}(T_i) + \left(\frac{1}{k+1}\right)\sum_j n_{ij}^2\,\text{var}(B_j) - 2\,\text{cov}(T_i, \left(\frac{1}{k+1}\right)\sum_j n_{ij}^2 B_j$$

$$= \left(\frac{rk}{k+1}\right)\sigma^2 \qquad (3.22)$$

Where:

$$\mathrm{var}\,(T_i) = r\sigma^2$$

$$\mathrm{var}(B_j) = (k+1)\sigma^2$$

$$\mathrm{cov}(T_i, B_j) = \frac{r}{k+1}\sigma^2$$

$$\mathrm{var}\,(Q_0) = \mathrm{var}\left(\frac{bk}{k+1}t_0 - \frac{r}{k+1}\sum_{u\neq 0} t_u\right)$$

$$\mathrm{var}(Q_0) = \left(\frac{bk}{k+1}\right)\sigma^2$$

$$\mathrm{var}(\hat{t}_i) = \left(\frac{(k+1)(v-1)^2}{vr(vk-1)}\right)\sigma^2$$

$$\mathrm{var}\,(\hat{t}_i - \hat{t}_k) = \mathrm{var}(\hat{t}_i) + \mathrm{var}\,(\hat{t}_k) - 2\,\mathrm{cov}(\hat{t}_i, \hat{t}_k)$$

Which reduced after some algebra to

$$\left(\frac{2(v-1)(k+1)}{r(vk-1)}\right)\sigma^2 \qquad (3.23)$$

$$\mathrm{var}\,(\hat{t}_i - \hat{t}_0) = \mathrm{var}\,(\hat{t}_i) + \mathrm{var}\,(\hat{t}_0) - 2\,\mathrm{cov}(\hat{t}_i, \hat{t}_0)\ ,$$

$$= \left(\frac{(k+1)(k+v-2)}{r(vk-1)}\right)\sigma^2 \qquad (3.24)$$

Inter-block information can also recovered, if the β_j are random variables distributed with zero means and variance σ_b^2 , they are further assumed to be uncorrelated, and independent of the e_{ij} . Under these circumstances the block totals can be regarded as a set of observation:

$$B_j = \sum_{i=0}^{v} n_{ij} t_i + \left[(k+1)\beta_j + \sum_i n_{ij} e_{ij}\right] \qquad (3.25)$$

The inter-block estimated are obtained by minimizing

$$\sum_j (\beta_j - \sum_{i=0}^{v} n_{ij}\,\tilde{t}_i)^2 \qquad (3.26)$$

With respect to $t_i \quad (i = 0,1,2,\ldots\ldots,v)$, where $\tilde{t}_i$ denote the inter-block estimates. The normal equations for inter-block estimation are:

$$\sum_{j=1}^{b} n_{ij} B_j = r\hat{t}_i + \lambda \sum_u \tilde{t}_u \qquad ; \qquad (u \neq i)$$

$$\sum_{j=1}^{b} n_{0j} B_j = b\tilde{t}_0 + r\sum_i \tilde{t}_i \qquad (3.27)$$

$$\sum_{j=1}^{b} n_{ij} B_j = r\tilde{t}_i + \lambda \sum_u \tilde{t}_u - \lambda \tilde{t}_i + r\tilde{t}_0$$

$$= r\tilde{t}_0 + (r-\lambda)\tilde{t}_i + \lambda \sum_u \tilde{t}_u \qquad (3.28)$$

$$E(B_j) = \sum_{i=0}^{v} n_{ij} t_i \qquad (3.29)$$

$$\operatorname{var}(\underline{B}) = (k+1)\left[\sigma^2 + (k+1)\sigma_b^2\right] I_b$$

Where I_b is identity matrix of order b.

$$\operatorname{cov}(B_j, B_{j'}) = zero \quad \text{For } j \neq j' \qquad (3.30)$$

The inter-block normal equations are: $\underline{Q}_1 = C_1 \underline{\tilde{t}}$ (3.31)

Where: $\underline{Q}_1 = \dfrac{1}{k+1} N \underline{B}$ (3.32)

$$C_1 = \frac{1}{k+1} N N' \qquad (3.33)$$

$$\underline{\tilde{t}}' = \tilde{t}_0, \tilde{t}_1, \ldots, \tilde{t}_v \qquad (3.34)$$

A solution for (3.21) is:

$$\underline{\tilde{t}} = (NN')^{-1} N \underline{B} \qquad (3.35)$$

Where: $NN' = \begin{bmatrix} b & rE_{1v} \\ rE_{v1} & (r-\lambda)I_v + \lambda E_{vv} \end{bmatrix}$ (3.36)

$$(NN')^{-1} = \frac{1}{M} \begin{bmatrix} (r-\lambda) & -rE_{1v} \\ -rE_{1v} & \left(\dfrac{M+r^2}{r-\lambda}\right) I_v \end{bmatrix} \qquad (3.37)$$

Where $M = b(r-\lambda) - vr^2$ (3.38)

For the inter-block the treatment parameter estimates can be obtained from (3.38) these are

$$\tilde{t}_0 = \frac{P_0(r-\lambda) - r\sum_{i=1}^{v} P_i}{M} \qquad (3.39)$$

$$\tilde{t}_0 = \frac{P_0(2r-\lambda) - r\,g\left[(k+1)\right]}{M}$$

Using the fact that: $\sum_{i=0}^{v} Q_i = g - (\frac{1}{k+1}) \sum_{i=0}^{v} P_i \Rightarrow g(k+1) = P_0 + \sum_{i=1}^{v} P_i$ (3.40)

$$\tilde{t}_i = \frac{(M + r^2)P_i - P_0[r(r-\lambda)]}{(r-\lambda)M} \qquad (3.41)$$

After using (3.41)

Where: $p_i = \sum_j n_{ij} B_j$ (3.42)

$$p_0 = \sum_j n_{0j} B_j \qquad (3.43)$$

$$g = \text{grand total of all observations} \qquad (3.44)$$

Then

$$\tilde{t}_i - \tilde{t}_0 = \frac{p_i M - p_0 (2r-\lambda)^2 + g\, r(k+1)](2r-\lambda)}{(r-\lambda)M} \qquad (3.45)$$

$$\text{var}(\tilde{t}_i - \tilde{t}_0) = \frac{(k+1)(\sigma^2 + (k+1)\sigma_b^2)[M + (2r-\lambda)^2]}{(r-\lambda)M}$$

$$= b_i \{\sigma^2 + (k+1)\sigma_b^2\}, b_i = \frac{[M + (2r-\lambda)^2]}{(r-\lambda)M}$$

(3.46)

$$\text{var}(\tilde{t}_i - \tilde{t}_u) = \frac{2(k+1)[\sigma^2 + (k+1)\sigma_b^2]}{(r-\lambda)} \qquad (3.47)$$

From the general theory of recovery of inter-block information as given by (RAO-1947) the normal equation for combined inter and intra-block estimates are

$$w\underline{Q} + w_1 \underline{Q}_1 = (wc + w_1 c_1)\underline{t}^* \qquad (3.48)$$

A solution for this is: $\underline{t}^* = (wc + w_1 c_1)^{-1}(w\underline{Q} + w_1 \underline{Q}_1)$ (3.49)

Where $w = \frac{1}{\sigma^2}$; $w_1 = \frac{1}{[\sigma^2 + (k+1)\sigma_b^2]}$ (3.50)

From (3.54) we obtain:

$$w\underline{Q} + w_1 \underline{Q}_1 = \{wc + (k+1)^{-1} w_1 NN'\}\underline{t}^* \qquad (3.51)$$

An element by element comparison of the matrix C of (3.8) with the matrix

$$wc + (k+1)^{-1} w_1 NN' \qquad (3.52)$$

Shows that (3.51) can be obtained from (3.8) by changing:

$$r_0 \to w r_0$$
$$b \to (w - w_1) b$$
$$\underline{Q} \to w\underline{Q} + w_1 \underline{Q}_1$$
$$r \to (w - w_1) r$$
$$\lambda \to (w - w_1)\lambda \qquad (3.53)$$

If we make this change in the solution of (3.8) we shall get a solution of (3.52), we note:
$bk = (k+1)b - b = (k+1)r_0 - b$, there fore

$$t_0^* = \frac{(k+1)(wQ_0 + w_1 Q_{01})}{(k+1)wr_0 - b(w - w_1)} \qquad (3.54)$$

$$t_i^* = \frac{(k+1)(wQ_i + w_1 Q_{i1})}{(rk+\lambda)(w - w_1)} + \frac{r(k+1)(w - w_1)((wQ_0 + w_1 Q_{01})}{(k+1)wr_0 - b(w-w_1)(rk+\lambda)(w-w_1)}$$

(3.55)

$$t_i^* - t_0^* = \frac{(k+1)(wQ_0 + w_1Q_{01})(w-w_1)(r-rk-\lambda)}{[(k+1)wr_0 - b(w-w_1)][(rk+\lambda)(w-w_1)]} + \frac{(k+1)(wQ_i + w_1Q_{i1})}{(rk+\lambda)(w-w_1)} \qquad (3.56)$$

$$t_i^* - t_u^* = \frac{(k+1)w(Q_i + Q_u) + (k+1)w_1(Q_{i1} - Q_{u1})}{(rk+\lambda)(w-w_1)} \qquad (3.57)$$

The variance of $t_i^* - t_0^*$ and $t_i^* - t_u^*$ are

$$\mathrm{var}(t_i^* - t_0^*) = \frac{(k+1)\big((k+1)wr_0 + (w-w_1)(r-rk-\lambda-b)\big)}{[(k+1)wr_0 - b(w-w_1)][(rk+\lambda)(w-w_1)]} = \frac{(k+1)\big((k+1)wr_0 - (w-w_1)(\lambda v + b)\big)}{[(k+1)wr_0 - b(w-w_1)](rk+\lambda)(w-w_1)} \qquad (3.58)$$

$$\mathrm{var}(t_i^* - t_u^*) = \frac{2(k+1)}{(rk+\lambda)(w-w_1)} \qquad (3.59)$$

Analysis of Variance (ANOVA)

Since the block and treatments are not orthogonal to each other there are two analysis of variance table. Table [1.a] gives the ANOVA for treatment adjusted for block while table [1.b] gives the ANOVA for blocks adjusted for treatments. The unadjusted blocks sum of squares (S.S.) can be split into two parts (1) The treatment component and (2) The intra-block error component. This decomposition stems from the inter-block mode, (Khirsagar1973), Raghavarao (1971). The sum of squares (S.S.) for treatments adjusted for blocks is given by: $\hat{t}_0 Q_0 + \sum_{i=1}^{v} \hat{t}_i Q_i$

(4.1)

Substituting the estimate of $\hat{t}_i$ and $\hat{t}_0$ from (3.18) and (3.19) in (4.1)

$$\left(\frac{(k+1)}{bk}\right)Q_0^2 + \sum_{i=1}^{v}\left(\left(\frac{(k+1)}{rk+\lambda}\right)(Q_i + \frac{1}{v}Q_0)\right)Q_i$$

$$= \left(\frac{(k+1)}{bk}\right)Q_0^2 + \left(\frac{k+1}{rk+\lambda}\right)\sum_{i=1}^{v}\left((Q_i^2) + \frac{1}{v}\left(\frac{k+1}{rk+\lambda}\right)\sum_{i=1}^{v} Q_0 Q_i\right) \qquad (4.2)$$

Using the fact that: $\sum_{i=1}^{v} Q_i = 0$,

$$\sum_{i=1}^{v} Q_0 Q_i = Q_0[Q_1 + \ldots + Q_v + Q_0 - Q_0] = Q_0[\sum_{i=1}^{v} Qi - Q_0] = -Q_0^2$$ There fore

$$\left(\frac{(k+1)}{bk}\right)Q_0^2 + \left(\frac{k+1}{rk+\lambda}\right)\sum_{i=1}^{v}(Q_i^2) + \frac{1}{v}\left(\left(\frac{k+1}{rk+\lambda}\right)\sum_{i=1}^{v} Q_0 Q_i\right) = \left(\frac{k+1}{rk+\lambda}\right)\left(\sum_{i=1}^{v}(Q_i^2) + \frac{\lambda}{r}Q_0^2\right) \qquad (4.3)$$

The sum of squares for blocks adjusted for treatment equals:

$$\frac{1}{k+1}\underline{B}'\underline{B} + \frac{k+1}{rk+\lambda}\left(\sum_{i=1}^{v} Q_i^2 + \tfrac{\lambda}{r} Q_0^2\right) - \tfrac{1}{r}\underline{T}'\underline{T} = \frac{1}{k+1}\sum_j B_j^2 + \frac{k+1}{rk+\lambda}\left(\sum_{i=1}^{v} Q_i^2 + \frac{\lambda}{r}Q_0^2\right) - \tfrac{1}{r}\sum_i T_i^2 \qquad (4.4)$$

For the model (3.29) the total sum of squares is: $\frac{1}{k+1}\sum_{j}^{b} B^2$ The inter-block error sum of squares is therefore is:

$$\frac{1}{k+1}\underline{B}'\underline{B} - \frac{1}{k+1}\underline{\tilde{t}}\,Q\,\underline{\tilde{t}} = \left(\frac{1}{k+1}\underline{B}'\underline{B} - \frac{g^2}{b(k+1)}\right) - \left(\frac{1}{k+1}\underline{\tilde{t}}\,Q_1\,\underline{\tilde{t}} - \frac{g^2}{b(k+1)}\right) \qquad (4.5)$$

Table (4.1) ANOVA table for testing treatment effects

Source of variation	Degrees of freedom	Sum of squares
Blocks unadjusted	b – 1	$\frac{1}{k+1}\underline{B}'\underline{B} - \frac{G^2}{b(k+1)}$
Treatment adjusted	v	$\frac{k+1}{rk+\lambda}\left(\sum Q_i^2 + \frac{\lambda}{v}Q_0^2\right)$
Intra block error	n – b – v	f E_e
Total	n – 1	$\underline{Y}'\underline{Y} - \frac{G^2}{b(k+1)}$

Table (4.2) ANOVA table for testing block effects

Source of variation	Degrees of freedom	Sum of squares
Blocks adjusted	b – 1	$(b-1)E_b$
Treatment unadjusted	v	$\frac{T_o^2}{r} + \sum \frac{T_i^2}{r} - \frac{G^2}{b(k+1)}$
Intra block error	f = n – b – v	f E_e
Total	n – 1	$\underline{Y}'\underline{Y} - \frac{G^2}{b(k+1)}$

Expected Values of Mean Squares in the ANOVA

From the model (3.1) $B_j s$ are NI $(0, \sigma_b^2)$ we obtain after some algebra the following expected values, Schffe's method of obtaining the expectation of this sum of squares is to consider for convenience the special case in which μ and T_i are all zero because they do not appear in the result. Then B_j, Q_i and T_i all have zero means, so that the expected values of their squares are equal to their variances, giving

$$\begin{aligned} &\operatorname{var}(B_j) = (k+1)^2\sigma_b^2 + (k+1)\sigma^2 \\ &\operatorname{var}(Q_i) = (r\,k/k+1)\sigma^2 \\ &\operatorname{var}(T_i) = r(\sigma^2 + \sigma_b^2)\ v \\ &\operatorname{var}(G) = b\operatorname{var}(B_j) = b[(k+1)^2\sigma_b^2 + (k+1)\sigma^2 \end{aligned} \qquad (5.1)$$

Then: $T = \left(\frac{T_0^2}{r_0} + \sum_{i=1}^{v}\frac{T_i^2}{r} - \frac{g^2}{b(k+1)}\right)$

$$E(T) = v\sigma^2 + (v-k)\sigma_b^2 \quad (5.2)$$

E(Block S. S. unadjusted = $\frac{b\,\mathrm{var}(B_j)}{k+1} + \frac{v(k+1)\,\mathrm{var}(Q_i)}{rk+\lambda} + \frac{v\,\mathrm{var}(T_i)}{r}$

(5.3)

$$= (b-1)\sigma^2 + \left(v(r-1) + (b-1)\sigma_b^2\right)$$

(5.4)

$$(fE_e) = \underline{Y}'\underline{Y} - \left[\frac{T_0^2}{r_0} + \sum_{i=1}^{v}\frac{T_i^2}{r}\right] - (b-1)E_b = v(r-1)\sigma^2$$

(5.5)

$$E(treat.adj.) = v\sigma^2 + \underline{t}'c\underline{t}$$

(5.6)

$$Total = \underline{Y}'\underline{Y} - \frac{G^2}{b(k+1)}$$

$$E(Total) = (bk+b-1)\sigma^2 + (k+1)(b-1)\sigma_b^2 \quad (5.7)$$

Writing E_e and E_b for the mean squares for the intra-block error and for block adjusted for treatments respectively. Solving for σ^2 and σ_b^2 and substituting in $\hat{w}$ and $\hat{w}_1$ the following estimates are obtained:

$$\hat{w} = \frac{1}{E_e};\hat{\sigma}_b^2 = \frac{(b-1)}{N-v}(E_b - E_e);w_1 = \frac{(E_b-E_e)(b-1)}{v(k+1)(b-1)E_b + (v-k)(b-v)E_e} \quad (5.8)$$

Table (5.1) Formulae for estimating treatment effects, differences & variances

Intra – block equations
$\hat{t}_0 = \frac{k+1}{bk}Q_0$
$\hat{t}_i = \frac{k+1}{rk+\lambda}\left(Q_i + \frac{Q_0}{v}\right)$
$\hat{t}_i - \hat{t}_0 = \frac{k+1}{rk+\lambda}\left(Q_i - \frac{\lambda}{r}Q_0\right)$
$\mathrm{var}(\hat{t}_i - \hat{t}_0) = \frac{(k+1)(k+v-2)}{r(vk-1)}\sigma^2$
Inter – block equations
$\tilde{t}_0 = \frac{P_0(r-\lambda) - r\sum_{i=1}^{v}P_i}{M}$

$$\tilde{t}_i = \frac{(M + r^2)P_i - P_0[r(r-\lambda)]}{(r-\lambda)M}$$

$$\tilde{t}_i - \tilde{t}_0 = \frac{p_i M - p_0(2r-\lambda)^2 + g\,r\,(k+1)](2r-\lambda)}{(r-\lambda)M}$$

$$\operatorname{var}(\tilde{t}_i - \tilde{t}_0) = b_i\,\{\sigma^2 + (k+1)\sigma_b^2\}, b_i = \frac{[M + (2r-\lambda)^2]}{(r-\lambda)M}$$

Combined inter & intra – block equations

$$t_0^* = \frac{(k+1)(wQ_0 + w_1 Q_{01})}{(k+1)wr_0 - b(w-w_1)}$$

$$t_i^* = \frac{(k+1)(wQ_i + w_1 Q_{i1})}{(rk+\lambda)(w-w_1)} + \frac{r(k+1)(w-w_1)((wQ_0 + w_1 Q_{01})}{(k+1)wr_0 - b(w-w_1)(rk+\lambda)(w-w_1)}$$

$$t_i^* - t_0^* = \frac{(k+1)(wQ_0 + w_1 Q_{01})(w-w_1)(r - rk - \lambda)}{[(k+1)wr_0 - b(w-w_1)][(rk+\lambda)(w-w_1)]} + \frac{(k+1)(wQ_i + w_1 Q_{i1})}{(rk+\lambda)(w-w_1)}$$

$$\operatorname{var}(t_i^* - t_0^*) = \frac{(k+1)\left((k+1)wr_0 + (w-w_1)(r - rk - \lambda - b)\right)}{[(k+1)wr_0 - b(w-w_1)][(rk+\lambda)(w-w_1)]} = \frac{(k+1)\left((k+1)wr_0 - (w-w_1)(\lambda\, v + b)\right)}{[(k+1)wr_0 - b(w-w_1)](rk+\lambda)(w-w_1)}$$

Example

Consider an experiment in which 6 treatments of varying levels of the protein riboflavin and total food intake were to be compared in a nutrition experiment on rats, where level **0** indicates the control. It was possible to obtain litters with 4 male rats each. A balanced incomplete block design of $(v+1)=6$ treatments in $b=10$ blocks of size $(k+1)=4$ rats was used.

Table (6) Code data for the example

(0) 2	(1) 2	(2) 3	(3) 6	$B_1 = 13$
(0) 4	(1) 5	(2) 3	(5) 7	$B_2 = 19$
(0) 1	(1) 4	(4) 8	(5) 7	$B_3 = 20$
(0) 4	(2) 3	(3) 10	(4) 11	$B_4 = 28$
(0) 10	(3) 1	(4) 8	(5) 9	$B_5 = 28$
(0) 3	(1) 1	(2) 5	(4) 5	$B_6 = 14$
(0) 3	(1) 2	(3) 6	(4) 10	$B_7 = 21$

(0) 5	(1) 7	(3) 5	(5) 8	$B_8 = 25$
(0) 2	(2) 9	(3) 10	(5) 11	$B_9 = 32$
(0) 7	(2) 3	(4) 12	(5) 14	$B_{10} = 36$

Each test treatment was replicated $r = 6$ times except the control which is replicated $r_0 = 10$ times. The variable measured was micrograms of riboflavin per 100ml of blood serum. The coded data is given in table (6). Treatment numbers are given in the upper right hand corner of each cell from (5.8) we get:

$\hat{w} = 0.155560; \hat{w}_1 = 0.0206$; The information from the three models (inter – block, intra – block and combined inter & intra – block) are summarized in table (5.1). For the data in the example, the treatment estimates, elementary treatment contrasts and their variances are summarized in table (6.a), (6.b) and (6.c).

From the equation: $Q_i = T_i - \frac{1}{k+1}\sum n_{ij} B_j$ we can calculate Q_0, Q_1, Q5; and from the equation $Q_{1i} = \frac{\sum n_{ij} B_j - r(k+1)\overline{G}}{r-\lambda}$; $\overline{G} = \frac{236}{40} = 5.9$ so that $r(k+1)\overline{G} = 141.6$; $r_0(k+1)\overline{G} = 236$ the efficiency ($E = \frac{v\lambda}{rk} = \frac{5\times 3}{6\times 3} = 0.83 = 83\%$

$T_0 = 41$	$T_1 = 21$	$T_2 = 26$	$T_3 = 38$	$T_4 = 54$	$T_5 = 56$	$\sum B_j = \sum T_i = 236$
$Q_0 = -18$	$Q_1 = -7$	$Q_2 = -9.5$	$Q_3 = 1.25$	$Q_4 = 17.25$	$Q_5 = 16$	$\sum_i Q_i = 0$
$Q_{10} = 0$	$Q_{11} = -9.866$	$Q_{12} = 0.133$	$Q_{13} = 1.8$	$Q_{14} = 1.8$	$Q_{15} = 6.133$	$\sum_i Q_{1i} = 0$

Table (6.1)
ANOVA Table for Testing Treatment Effects in the Example

Source of variation	Degrees of freedom	Sum of squares	Mean Square
Block unadjusted	9	127.6	14.178
Treatment adjusted	5	169.2905	
Intra-block error	25	160.7095	6.4283
Total	39	457.6	

Table (6. 2) ANOVA Table for Testing Block Effects in the Example

Source of variation	Degrees of freedom	Sum of squares	Mean square error
Block adjusted	9	85.6905	9.521
Treatment unadjusted ununununadjusted	5	211.2	
Intra-block error	25	160.7095	6.4283
Total	39	457.6	

Table (6.a) Intra-block model

$\hat{t}_o$	-2.4	$\hat{t}_1 - \hat{t}_o$	0.381
$\hat{t}_1$	-2.0	$\hat{t}_2 - \hat{t}_o$	-0.095
$\hat{t}_2$	-2.5	$\hat{t}_3 - \hat{t}_o$	1.952
$\hat{t}_3$	-0.5	$\hat{t}_4 - \hat{t}_o$	5.00
$\hat{t}_4$	2.6	$\hat{t}_5 - \hat{t}_o$	4.70
$\hat{t}_5$	2.9		

$$var(\hat{t}_i - \hat{t}_0) = 1.8366$$

Table (6.b) Inter-block model

$\tilde{t}_o$	23.6	$\tilde{t}_1 - \tilde{t}_o$	-33.47
$\tilde{t}_1$	-9.87	$\tilde{t}_2 - \tilde{t}_o$	-23.47
$\tilde{t}_2$	0.13	$\tilde{t}_3 - \tilde{t}_o$	-21.80
$\tilde{t}_3$	1.80	$\tilde{t}_4 - \tilde{t}_o$	-21.80
$\tilde{t}_4$	1.80	$\tilde{t}_i - \tilde{t}_o$	-17.47
$\tilde{t}_5$	6.13		

$$var(\tilde{t}_i - \tilde{t}_0) = 10.6085$$

Table (6.c)Combined inter - & intra block equations

t_0^*	-2.3	$t_1^* - t_0^*$	-0.3

t_1^*	-2.6	$t_2^* - t_0^*$	-0.5
t_2^*	-2.8	$t_3^* - t_0^*$	1.7
t_3^*	-0.6	$t_4^* - t_0^*$	5.4
t_4^*	3.1	$t_5^* - t_0^*$	5.2
t_5^*	2.4		

$$var(t_i^* - t_0^*) = 0.825$$

In this example we dealt with the estimation of weight for combining inter and intra-block estimates, and we consider the weight for combining inter-block estimates of treatment contrast. The advantage of this study is to get a minimum variance of the treatment contrast between the inter and intra block estimates when a control treatment is added to each block in B. I. B. design. From tables (6.a) and (6.c) we conclude that treatment five is more effective than other treatments when compared to the control. The reduction in variance due to recovery of inter-block information is however very small.

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مخطط القوالب الناقصة المتوازنة في وجود معالجة ضابطة

سوسن حسب الرسول بابكر محمد ـ أستاذ مساعد
رئيس قسم الرياضيات والفيزياء- كلية التربية-جامعة الجزيرة– ود مدني- السودان- ص. ب. 20

الملخص

تم تطوير نظرية تصميم وتحليل التجارب بشكل أساسي بواسطة الإحصائيين العاملين في مجال البحوث الزراعية؛ وقد وجدت هذه النظرية الآن تطبيقات في مجالات أخرى من الأبحاث وذلك لأنها تبنى على المبادئ العامة التي تتعلق بالسلوك الإحصائي. خاصة الملاحظات التي تنجم إما اختيارياً في الطبيعة أو داخل المعامل. هذا النموذج الإحصائي يستخدم عندما يكون عدد المعالجات كبيراً بحيث يصعب جعل القوالب متجانسة، وتصمم التجربة بحيث تحتوي القوالب الناقصة على عدد محدد من المعالجات؛ وكل معالجتين تظهران معاً في القالب الواحد نفس العدد من المرات والغرض من التصميم إبعاد آثار اختلافات المعالجات عن خطأ التجريب وتوحيد الأخطاء المعيارية للمقارنات. هذه المفاهيم قدمها أول مرة *(Das 1954)* ثم *(Pearce 1960)*. أما *(Pesek 1974)* فقد اهتم بدراسة هذا التصميم بزيادة المعالجات الضابطة في كل قالب كما عمل على توظيف المعادلة العامة ل *(Rao)* للحصول على التباين في المعالجة الأولية بالمقارنة بين أي زوجين من المعالجات التي يراد اختبارها والمعالجة الضابطة وبين أن التصميم أكثر كفاءة من مخطط القوالب الناقصة المتوازنة عند مقارنة المعالجات مع المعالجة الضابطة لكنه أقل كفاءة عند المقارنات الثنائية بين المعالجات المراد اختبارها. في هذه الدراسة اهتمت الباحثة بدراسة الحالة عند إضافة معالجة ضابطة لكل قالب في هذا التصميم وتم الحصول على تباين المعالجات الأولية بين أي زوجين من المعالجات المراد اختبارها وبين المعالجة الضابطة؛ كما تم التحقق من النظرية العامة لأثر تقديرات معالجات القوالب من الداخل عند اضافة المعالجة الضابطة للتصميم.

هذا التصميم يمكن أن يستخدم في المجال الطبي في اكتشاف فاعلية العقاقير الجديدة والأمراض المستعصية وفي المجال الاقتصادي والعديد من المجالات الأخرى.