

Geometrical Effects of Conic Sections on the Motion of Bodies

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ABSTRACT

The subject of dynamics is generally divided into two branches; kinematics which is concerned with the geometry of motion apart from all considerations of force, mass or energy. And kinetics which is concerned with the effects of forces on the motion of bodies. In this paper, we explained the basic geometrical concepts of the conic sections (ellipse, parabola, and hyperbola) and their effects on the suitable kind of motion and the orbits of bodies.

INTRODUCTION

In order to describe the motion of a body or of a point, two things are needed, the frame of reference and the time-keeper. It is not possible to describe absolute motion, but only motion relative to surrounding objects; and suitable frame of reference depends on the kind of motion that is desired to describe. Thus if the motion is rectilinear the distance from a fixed point on the line is a sufficient description of the position of the moving point; and in more general cases systems of two or of three rectangular axes may be chosen as a frame of reference. For example, in the case of a body projected from the surface of the Earth a set of axes with the origin at the point of projection would be suitable for the description of motion relative to the Earth. But, for the description of the motion of the planets, it would be more convenient to take a frame of the axes with an origin at the sun's Centre.

It is important to realize that there is no such thing as absolute time, but the period of rotation of the Earth relative to the fixed stars provides a unit of time, the sidereal day, which, so far as it can be tested with other time measures, is constant and therefore adequate for the purposes of ordinary dynamics.

The functions involved in dynamical problems are for the most part differential coefficients with regard to time (t) as the independent variable. Thus motion is change of position or displacement, velocity is rate of displacement and acceleration is rate of change of velocity. Hence, if x denotes a distance, dx/dt denotes a velocity and d^2x/dt^2 denotes an acceleration. The formulation of a dynamical problem therefore in general consists of one or more relations between certain variables (coordinates of position) and their differential coefficients with regard to time are called differential equations [1].

Definitions of Conic Sections

Geometric figures can be obtained by intersecting a double-napped right circular cone with a plane, called conic sections. By varying the position of the plane (the intersecting angle), we obtain an ellipse, a parabola, or a hyperbola.

An **ellipse** is the set of all points in a plane, the sum of whose distances from two fixed points (the foci) in the plane is constant [3].

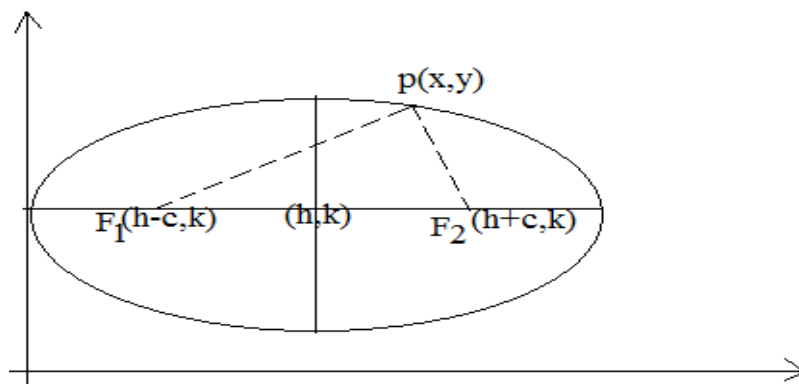


Figure (1): An ellipse of Center (h, k) and foci F_1, F_2 .

One of the standard forms:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \quad a^2 = b^2 + c^2.$$

Which is represent a circle when $a = b$.

A **parabola** is the set of all points in a plane equidistant from a fixed point F (the focus) and a fixed line l (the directrix) in the plane [3].

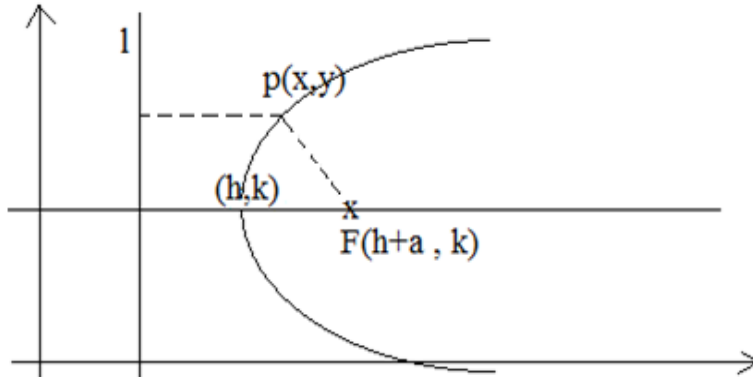


Figure (2): A parabola of vertex (h , k) and focus F.

One of the standard forms: $(y - k)^2 = 4a(x - h)$.

A **hyperbola** is the set of all points in a plane, the difference whose distances from two fixed points (the foci) in the plane is a positive constant [3].

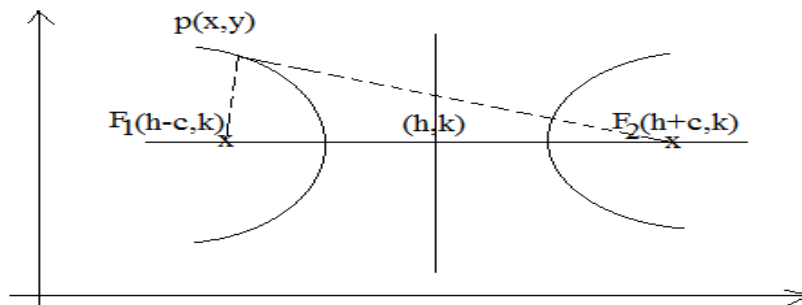


Figure (3): A hyperbola of centre (h , k) and foci F_1, F_2 .

One of the standard forms:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1, \quad c^2 = a^2 + b^2.$$

Velocity and Acceleration in Polar Coordinate

Let the position of a point P be defined by its distance r from a fixed origin O and the angle θ that OP makes with a fixed axis Ox. The cartesian coordinates (x, y) of P are connected with the polar coordinates (r, θ) by the relations:

$$x = r \cos \theta, \quad y = r \sin \theta.$$

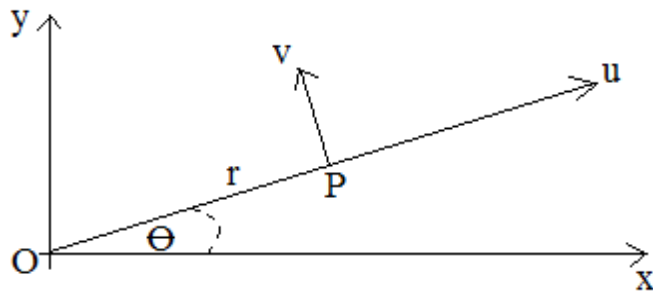


Figure (4): Components of Velocity of P.

Let u, v denote the components of velocity of P in the direction OP and at right angles to OP in the sense in which θ increases. The resultant of the components u, v is also the resultant of the components of \dot{x}, \dot{y} .

Therefore by resolving parallel to Ox and Oy we get:

$$u \cos \theta - v \sin \theta = \dot{x} = \frac{d}{dt}(r \cos \theta) = \dot{r} \cos \theta - r \dot{\theta} \sin \theta,$$

$$\text{and } u \sin \theta + v \cos \theta = \dot{y} = \frac{d}{dt}(r \sin \theta) = \dot{r} \sin \theta + r \dot{\theta} \cos \theta.$$

Solving these equations for u and v clearly gives:

$$u = \dot{r} \quad \text{and} \quad v = r \dot{\theta};$$

and these are the polar components of velocity [2].

In like manner if f_1, f_2 denote the components of acceleration along and at right angle to OP, since these have the same resultant as :

$$\begin{aligned} \ddot{x}, \ddot{y} \text{ and } f_1 \cos \theta - f_2 \sin \theta &= \ddot{x} = \frac{d^2}{dt^2}(r \cos \theta) \\ &= (\ddot{r} - r\dot{\theta}^2) \cos \theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \sin \theta, \\ \text{and } f_1 \sin \theta + f_2 \cos \theta &= \ddot{y} = \frac{d^2}{dt^2}(r \sin \theta) \\ &= (\ddot{r} - r\dot{\theta}^2) \sin \theta + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \cos \theta \\ \Rightarrow f_1 &= \ddot{r} - r\dot{\theta}^2, \quad f_2 = r\ddot{\theta} + 2\dot{r}\dot{\theta}. \end{aligned}$$

These components constitute a third representation of the velocity and acceleration of a point moving in a plane; they are sometimes called radial and transverse components. Note that the transverse component of acceleration may also be written as:

$$\frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}).$$

RESULTS AND DISCUSSION

This sections for discussion all the suitable orbits (general, circular, elliptic, parabolic, and hyperbolic) to explain the geometrical effects of conic sections on motion of bodies.

2-1 General Orbits: Usually we denote the constant h by:

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$$h = r^2 \dot{\theta} = pv = x\dot{y} - y\dot{x}.$$

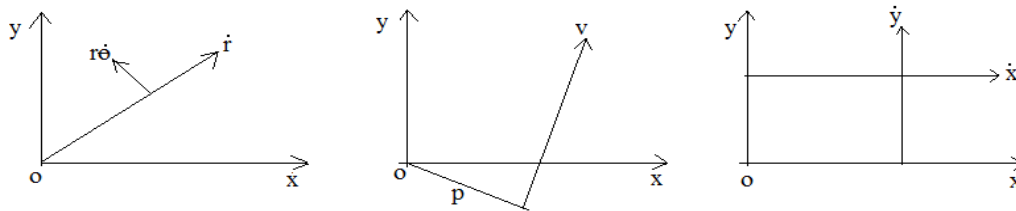


Figure (5): The moment of the velocity about a fixed point.

There are two common formulae for the law of force, one for use when the orbit is given by its (r, p) equation, where r denotes the radius vector and p the perpendicular from the origin to the tangent, and the other for use when the orbit is given by its polar equation (a relation between r and θ).

(i) Orbit given by (r, p) equation.

Let f denote the required force per unit mass, i.e. the acceleration towards the Centre O.

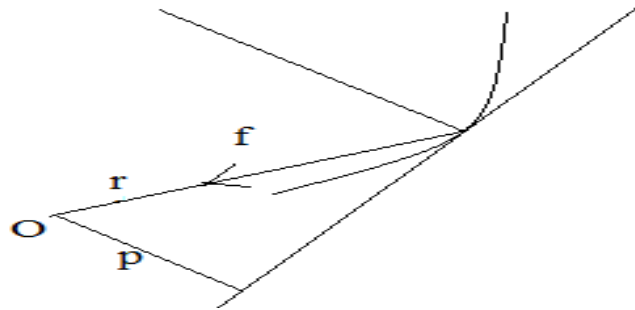


Figure (6): Orbit given by (r, p) equation.

If v is the velocity, the acceleration along the inward normal is v^2/ρ , resolving along the normal [4]

$$\frac{fv}{r} = \frac{v^2}{\rho}, \quad \rho = r dr / dp, \quad vp = h \Rightarrow f = \frac{rh^2}{p^3 \rho} = \frac{h^2}{p^3} \frac{dp}{dr}.$$

(ii) Orbit given by (r, θ) equation

If we write u for $1/r$, and make use of the formula:

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2, \quad -\frac{2}{p^3} \frac{dp}{dr} = \frac{d}{dr} \left(\frac{1}{p^2}\right) = \frac{d}{d\theta} \left\{ u^2 + \left(\frac{du}{d\theta}\right)^2 \right\} / \frac{dr}{d\theta}.$$

$$\Rightarrow f = h^2 u^2 \left\{ u + \frac{d^2 u}{d\theta^2} \right\}.$$

The converse problem – given the law of force to find the orbit – is solved by substituting the given expression for f in the previous formulae and then integrating the resulting equation, and provided that the orbit can be identified by its (r, p) equation it is clear that first formula simpler to use as the orbit will be found by a single integration.

Circular Orbits

Since a particle describing a circle with uniform velocity v has a constant acceleration v^2/a towards the Centre, where a is the radius, it follows that a particle can describe a circle under any constant force per unit mass f tending the Centre provided that it is projected at right angles to the radius with velocity \sqrt{af} .

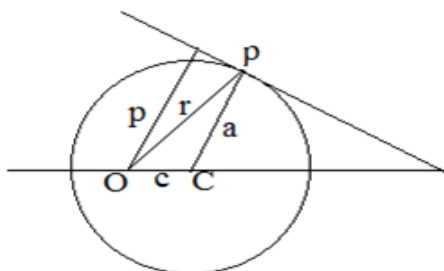


Figure (7)

$$\frac{a^2 b^2}{p^2} = a^2 + b^2 - r^2.$$

): Circle with centre C and any internal point O as centre of force.

It is easy to see from the triangle OCP that $c^2 = r^2 + a^2 - 2ap$, so that $dp/dr = r/a$. Therefore the force towards O under which the particle would describe the circle is

$$f = \frac{h^2}{p^2} \frac{dp}{dr} = \frac{h^2 r}{p^2 a}, \quad vp = h$$

$$\text{if } c = a, \quad r^2 = 2ap \Rightarrow f = \frac{8a^2 h^2}{r^5}, \quad v = \frac{2ah}{r^2}.$$

Conversely a particle projected from a point P at a distance r from O with velocity $\sqrt{\frac{1}{2}\mu / r^2}$ under the action of a force μ/r^5 per unit mass towards O will describe a circle passing through O, and the position of the Centre of the circle and therefore also its radius depends only on the direction of the projection [5].

Elliptic Orbits

Force directed to the Centre:

Let P be a point on the ellipse whose Centre is C and semi- axes a and b. If p is the central perpendicular on the tangent at P and CD is the radius conjugate to CP from the properties of the ellipse, we have

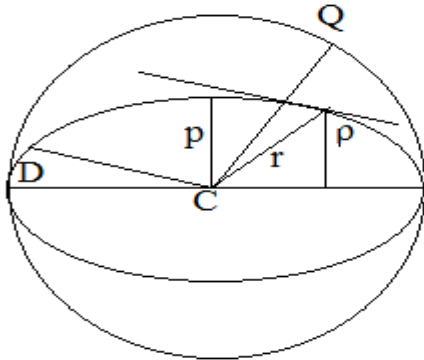


Figure (8): The Ellipse whose Centre C and semi-axes a and b .

This is the (r, p) equation of the ellipse when the Centre is the origin. By differentiating:

$$\frac{a^2 b^2}{p^2} \frac{dp}{dr} = r \Rightarrow f = \frac{h^2}{p^3} \frac{dp}{dr} = \frac{h^2}{a^2 b^2} r = \mu r, \quad h = ab\sqrt{\mu}.$$

Force directed to the foci:

Let S, H be the foci of an ellipse and SY, HZ the perpendiculars to the tangent at P. To find the (r, p) equation with S as origin we assume three properties of the ellipse:

- (i) $SP + HP = 2a$
- (ii) $SY \cdot HZ = b^2$,
- (iii) The tangent is equally inclined to the focal distances so that SPY, HPZ are singular triangles.

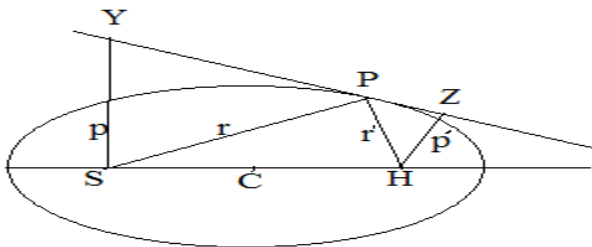


Figure (9): The Ellipse whose Centre C and Foci S, H.

If $SP = r$, $HP = r'$, $SY = p$ and $HZ = p'$

$$\frac{p}{r} = \frac{p'}{r'} , \sqrt{\frac{pp'}{rr'}} = \frac{b}{\sqrt{\{r(2a-r)\}}}$$

$$\frac{b}{p} = \sqrt{\left(\frac{2a-r}{r}\right)} \Rightarrow \frac{b^2}{p^2} = \frac{2a}{r} - 1$$

$$\Rightarrow f = \frac{h^2}{p^3} \frac{dp}{dr} = \frac{h^2 a}{b^2} \cdot \frac{1}{r^2} = \frac{\mu}{r^2} , v^2 = \frac{2\mu}{r} - \frac{\mu}{a}$$

2-4 Parabolic Orbits (Force directed to Focus)

The (r, p) equation of a parabola is obtained from the facts that if the tangent at P meets the tangent at the vertex A in Y and S is the focus, then SY is at right angles to PY and the triangles ASY, YSP are similar [2].

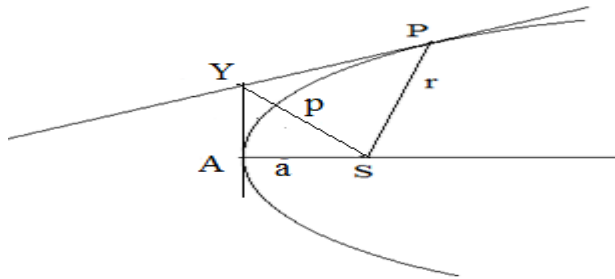


Figure (10): The Parabola whose focus S and vertex A.

This gives:

$$p^2 = ar \Rightarrow \frac{1}{p^2} = \frac{1}{ar} \Rightarrow \frac{2}{p^3} \frac{dp}{dr} = \frac{1}{ar^2}$$

Hence the force to the focus necessary for the description of the parabola is given by:

$$f = \frac{h^2}{p^3} \frac{dp}{dr} = \frac{h^2}{2a} \cdot \frac{1}{r^2} = \frac{\mu}{r^2}$$

where $\mu = \frac{h^2}{2a}$, or $h^2 = \mu \times \text{semi-latus rectum}$.

Also, if v is the velocity at P then:

$$v^2 = \frac{h^2}{p^2} = \frac{2a\mu}{ar} = \frac{2\mu}{r}$$

Hyperbolic Orbits (Force directed to the foci).

To find the (r, p) equation of a hyperbola we assume the corresponding as for the ellipse :

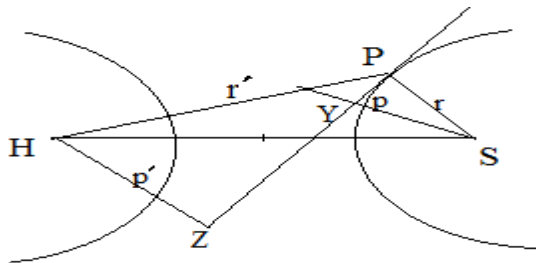


Figure (11): The Hyperbola whose foci H and S.

- (i) $SY.HZ = b^2$,
 - (ii) $r' - r = 2a$,
 - (iii) the tangent bisects the angle between the focal distances.
- Hence we have:

$$\frac{p}{r} = \frac{p'}{r'} = \sqrt{\frac{pp'}{rr'}} = \frac{b}{\sqrt{r(2a+r)}}, \text{ so that } \frac{b^2}{p^2} = \frac{2a}{r} + 1$$

$$\Rightarrow \frac{b^2}{p^2} \frac{dp}{dr} = \frac{a}{r^2} \Rightarrow f = \frac{h^2}{p^3} \frac{dp}{dr} = \frac{h^2 a}{b^2} \cdot \frac{1}{r^2} = \frac{\mu}{r^2}, \text{ where } \mu = h^2 a / b^2,$$

or $h^2 = \mu \ell$, if $\ell \equiv \text{semi-latus rectum } b^2 / a$.

In this case:

$$v^2 = \frac{h^2}{p^2} = \frac{h^2}{b^2} \left(\frac{2a}{r} + 1 \right) = \frac{\mu}{a} \left(\frac{2a}{r} + 1 \right), \text{ or } v^2 = \frac{2\mu}{r} + \frac{\mu}{a},$$

gives the velocity [1].

CONCLUSIONS

If a particle (body) is describing an orbit under the action of a force directed to a fixed point then we conclude that:

- (i) The orbit must be a plane curve; because at any instant the particle is moving in the plane through the tangent to its path and the fixed point and the only force acting on the particle lies in this plane, therefore the particle continues to move in this plane.
- (ii) The rate of description of area by radius vector drawn from the fixed point to the particle is constant; for there is no force at right angles to the radius vector so that the transverse component of acceleration is zero throughout the motion,

i.e. $\frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = 0 \Rightarrow r^2 \dot{\theta} = \text{const.}$

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التأثير الهندسي للقطع المخروطية على حركة الأجسام

د. عبد الرحيم بثير حامد

أستاذ مشارك ، قسم الرياضيات والفيزياء ، كلية التربية ، حنتوب ،

جامعة الجزيرة ، السودان .

ملخص

أنّ دراسة الديناميكا بصورة عامة تنقسم الى فرعين رئيسيين، الكينماتيكا والتي تهتم بدراسة هندسة الحركة للأجسام (المنظور الهندسي لحركة الأجسام) تحت تأثيرات القوة ، الكتلة ، الطاقة ، وغيرها. والكيناتيكا التي تهتم بتأثيرات القوى المختلفة على الأجسام. في هذه الورقة العلمية، تم توضيح المفاهيم الهندسية الأساسية للقطع المخروطية (القطع الناقص، القطع المكافئ، والقطع الزائد) وأثرها على حركة الأجسام ومساراتها التي تناسب طبيعة كل منها.