

## **Developed Upgraded Method for Solving Problems of Factor Groups**

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### **Abstract**

Through teaching the syllabus of the group theory, appeared some ambiguity in presentation the basic concepts of factor (quotient) groups. This is because the definitions and theorems expressed in abstract language, also the explanation examples are inadequate in most references. For this reason, and after research, the researcher has found a way to reach an upgraded developed method for understanding and solving the problems of finite and infinite factor groups.

## INTRODUCTION

This paper started by studying the normal subgroups, to introduce the factor groups.

### Definition:

If  $H$  is a subgroup of the group  $G$  and  $a \in G$ , then the set: **1:**  
 $aH = \{ah : h \in H\}$  operator element, is said to be left coset, and  $a$  called  
 similarly  $Ha$  called right coset (Burton, 1993).

Note that  $H$  is a left coset, since  $eH = H$  ( $e$  : Identity element).

either **Lemma 1:** If  $H$  is a subgroup of the group  $G$  and  $a, b \in G$ , then  
 $aH = bH$  or  $aH \cap bH = \phi$ .

**Proof:**

$$aH = \{ah : h \in H\}$$

### Definition

**2:**

left cosets is called If  $H$  is a subgroup of the group  $G$ , then the number of the distinct  
 (Rotman, 1990). the index of  $H$  in  $G$  and denoted by  $[G : H]$ ,

### Lagrange Theorem:

If  $H$  is a subgroup of the finite group  $G$  then,

$$|G| = [G : H] \cdot |H|$$

such that  $|G|$  and  $|H|$  are the orders of  $G$  and  $H$  respectively.

Note that the order of  $H$  divides the order of  $G$  (Dixon, 1977).

**Definition**

**3:**

If  $H$  is a subgroup of  $G$  then,  $H$  called normal subgroup of  $G$  if,

$$aH = Ha, \forall a \in G \text{ and denoted by } H\Delta G.$$

**Lemma**

**2:**

equivalent: If  $H$  is a subgroup of the group  $G$ , then the following statements are

(i)  $H\Delta G$

(ii)  $H = aHa^{-1} \forall a \in G$

(iii)  $aH = Ha \forall a \in G$

**Proof:**

(i)  $\Rightarrow$  (ii):

$$H\Delta G \Rightarrow aHa^{-1} \subseteq H \forall a \in G, \text{ let } h \in H \Rightarrow h = aa^{-1}haa^{-1}$$

$$\text{but } a^{-1}ha \in a^{-1}Ha, H\Delta G \Rightarrow a^{-1}Ha \subseteq H (h' = a^{-1}ha \text{ such that } h' \in H)$$

$$\text{Now } h = ah'a^{-1} \in aHa^{-1} \text{ or } aHa^{-1} \supseteq H.$$

(ii)  $\Rightarrow$  (i) is trivial.

(ii)  $\Rightarrow$  (iii):

$$\because aHa^{-1} = H$$

$$\Rightarrow Ha = (aHa^{-1})a$$

$$\begin{aligned} \text{but } (aHa^{-1})a &= \{(aha^{-1})a : h \in H\} \\ &= \{ah : h \in H\} \\ &= aH \end{aligned}$$

$$\Rightarrow aH = Ha$$

(iii)  $\Rightarrow$  (ii) :  $aH = Ha \Rightarrow a^{-1}(aH) = a^{-1}(Ha)$

$$\Rightarrow H = aHa^{-1} \forall a \in G.$$

**Definition**

**4:**

$H$  in  $G$ , which If  $H$  is a normal subgroup of the group  $G$  then, all the left cosets of  $H$  in  $G$  are denoted by  $G/H$  and the operation  $(\otimes)$  defined by:

$$(aH) \otimes (bH) = abH \forall a, b \in G$$

, called factor (quotient) group and denoted by  $(G/H, \otimes)$ .

(Macdonald, 1998). Note that all the elements of the factor group are left cosets

**Results:**

The main purpose of this paper is to derive developed suggested method to explain the complex concept of the factor groups via two easy types.

**Model 1** ( for finite groups):

For the system  $(G/H, \otimes)$ , consider  $G = \{1, 3, 5, 7\}$ ,  $H = \{1, 5\}$  and  $(\otimes)$  is to be the multiplication modulo 8. Firstly, find all the left cosets :

*to prove  $aH = Ha$  and  $H\Delta G$*

$$1 \otimes H = \{1 \otimes 1, 1 \otimes 5\} = \{1, 5\} = H \otimes 1$$

$$3 \otimes H = \{3 \otimes 1, 3 \otimes 5\} = \{3, 7\} = H \otimes 3$$

$$5 \otimes H = \{5 \otimes 1, 5 \otimes 5\} = \{5, 1\} = H \otimes 5$$

$$7 \otimes H = \{7 \otimes 1, 7 \otimes 5\} = \{7, 3\} = H \otimes 7$$

$\Rightarrow H\Delta G$  (*H is a normal subgroup of G*)

*Note that the index is 2 and Lagrange theorem verified , when*

$$|G| = [G : H] \cdot |H| \quad (4 = 2 \cdot 2).$$

The elements of  $(G/H, \otimes) = \{1\otimes H, 3\otimes H, 5\otimes H, 7\otimes H\}$  and then, from tables (1) and (2) , all the conditions of the group are valid.

Table 1: Operation table of the system  $(G/H, \otimes)$ .

| $\otimes$    | $1\otimes H$ | $3\otimes H$ | $5\otimes H$ | $7\otimes H$  |
|--------------|--------------|--------------|--------------|---------------|
| $1\otimes H$ | $1\otimes H$ | $\otimes H3$ | $\otimes H5$ | $7\otimes H$  |
| $3\otimes H$ | $\otimes H3$ | $1\otimes H$ | $7\otimes H$ | $\otimes H 5$ |
| $5\otimes H$ | $\otimes H5$ | $7\otimes H$ | $1\otimes H$ | $3\otimes H$  |
| $7\otimes H$ | $7\otimes H$ | $5\otimes H$ | $3\otimes H$ | $1\otimes H$  |

Table 2: The inverses of the elements of  $(G/H, \otimes)$ .

|    |               |               |               |               |
|----|---------------|---------------|---------------|---------------|
| a  | $1 \otimes H$ | $3 \otimes H$ | $5 \otimes H$ | $7 \otimes H$ |
| a' | $1 \otimes H$ | $3 \otimes H$ | $5 \otimes H$ | $7 \otimes H$ |

Note that the argument above is still valid if  $G/H$  is defined to be only the distinct left cosets  $\{1 \otimes H, 3 \otimes H\}$ .

**Model 2** ( for infinite groups):

Consider the system  $(\mathbb{Z}/3\mathbb{Z}, +)$  such that  $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ ,  $n\mathbb{Z} = \{\dots, -2n, -n, 0, n, 2n, \dots\}$ . Obviously  $n\mathbb{Z} \subseteq \mathbb{Z}$ , since  $\mathbb{Z}$  is abelian, we get  $n\mathbb{Z} \triangleleft \mathbb{Z}$  for  $n = 1, 2, 3, \dots$ .

Atypical left cosets is  $(a + n\mathbb{Z})$  for  $a \in \mathbb{Z}$ .

$\mathbb{Z}/n\mathbb{Z} = \{a + n\mathbb{Z} : a \in \mathbb{Z}\} = \{i + n\mathbb{Z} : 0 \leq i < n\} = \mathbb{Z}_n$ . The group operation in  $\mathbb{Z}/n\mathbb{Z}$  is  $(a + n\mathbb{Z}) + (b + n\mathbb{Z}) = (a + b) + n\mathbb{Z} = r + n\mathbb{Z}$  where  $a + b = qn + r$  and  $0 \leq r < n$ . Thus the operation in  $\mathbb{Z}/n\mathbb{Z}$  is identical to modular addition that performed in  $\mathbb{Z}_n$

e.g.  $\mathbb{Z}/3\mathbb{Z} = \{0 + 3\mathbb{Z}, 1 + 3\mathbb{Z}, 2 + 3\mathbb{Z}\}$  which has only 3 distinct cosets and infinite number of representatives for each of the cosets and thus an infinite number of equivalent ways to represent  $\mathbb{Z}/3\mathbb{Z}$ .

From tables (3) and (4), we can note that all the conditions of the group are valid.

Table 3: Operation table of the system  $(\mathbb{Z}/3\mathbb{Z}, +)$ .

| +                 | $0 + 3\mathbb{Z}$ | $1 + 3\mathbb{Z}$ | $2 + 3\mathbb{Z}$ |
|-------------------|-------------------|-------------------|-------------------|
| $0 + 3\mathbb{Z}$ | $0 + 3\mathbb{Z}$ | $1 + 3\mathbb{Z}$ | $2 + 3\mathbb{Z}$ |
| $1 + 3\mathbb{Z}$ | $1 + 3\mathbb{Z}$ | $2 + 3\mathbb{Z}$ | $0 + 3\mathbb{Z}$ |
| $2 + 3\mathbb{Z}$ | $2 + 3\mathbb{Z}$ | $0 + 3\mathbb{Z}$ | $1 + 3\mathbb{Z}$ |

Table 4: The Inverses of the elements of system  $(\mathbb{Z}/3\mathbb{Z}, +)$ .

|    |                   |                   |                   |
|----|-------------------|-------------------|-------------------|
| a  | $0 + 3\mathbb{Z}$ | $1 + 3\mathbb{Z}$ | $2 + 3\mathbb{Z}$ |
| a' | $0 + 3\mathbb{Z}$ | $2 + 3\mathbb{Z}$ | $1 + 3\mathbb{Z}$ |

Thus, the system  $(\mathbb{Z}/3\mathbb{Z}, +)$  represents factor (quotient) group.

**Conclusion:**

In this short research, developed and easiest way for explaining the complex concept of factor (quotient) groups, was suggested and represented by two upgraded solved problems.

## REFERENCES

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## طريقة مطورة ومطرودة التسلسل لحل مسائل الزمر الكسرية

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### المستخلص

من خلال تدريس مقرر نظرية الزمر ، ظهر بعض الغموض في عرض مفهوم الزمر الكسرية ، وذلك لأن التعريفات والمبرهنات يعبر عنها بلغة مجردة والأمثلة تكون غير كافية في أغلب المراجع. ولهذا السبب وبعد البحث والاطلاع ، تمّ التوصل لطريقة سهلة ومطورة في حل مسائل الزمر الكسرية من خلال نموذجين لمعالجة ذلك الغموض في كل من الزمر الكسرية المنتهية وغير المنتهية.