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Design of Control Systems in State Space and Solving Pole-Placement Problem with Four different Methods

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Abstract

This paper presenting a design method commonly called the pole-placement or pole-assignment technique. Assumed that all state variables are measurable and available for feedback. It will be shown that if the system considered is completely state controllable, then poles of the closed-loop system may be placed at any desired locations by means of state feedback through an appropriate state feedback gain matrix. The present design technique begins with a determination of the desired closed-loop poles based on the transient-response and/or frequency-response requirements.

The beginning with presenting the basic materials on pole-placement in regulator systems. Then the main problem solved with three different methods, followed by MAT LAB solution. Finally, the criteria of each method and the comparison between them discussed.

Basic Terminologies

State Space: The n-dimensional whose coordinate axes are X_l, $X_2,\ldots,X_n$

System: A combination of mathematical components that act together and perform a certain objective (Physical Phenomena, Biological, Economic, .. e), (Dorfman, 2003).

Feedback Control: Operation that in the presence of disturbances tends to reduce the different between the output of a system and some reference input and does so on the basis of this difference (Macki, 1995).

Closed-loop Control System: Feedback control system.

Open-loop Control System: That in which the output has no effect on the control action in order to reduce system error.

Pole: Zero point in which the system may be state controllable.

Transient Response: The time response in which control system goes from initial state to final state (Ogata, 2002).

Introduction

The pole-placement is somewhat similar to the root-locus method in that we place closed-loop poles at desired locations. The basic difference is that in the root-locus design we place only the dominant closed-loop poles at the desired locations, while in the pole-placement design we place all closed-loop poles at the desired locations.

Let us assume that we decide that the desired closed-loop poles are to be at $S = \mu_1$, $S = \mu_2,\ldots,S = \mu_n$. By choosing an appropriate gain matrix for state feedback,. it is possible to force the system to have closed-loop poles at the desired locations, provided the original system is completely state controllable (Ogata, 2002).

This paper aimed to discuss state-space design methods based on the pole-placement method and to solve pole-placement problem with different four methods including MAT LAB solution. Thus, we limit our discussions to single-input-single-output systems. That is, we assume the control signal u(t) and output signal y(t) to be scalars. In the derivation we assume that the reference input r(t) is zero. In what follows we shall prove that a necessary and sufficient condition that the closed-loop poles can be placed at any arbitrary locations in the s plane is that the system be completely state controllable. Then we shall discuss methods for determining the required state feedback gain matrix (Green, 2007).

Design by Pole-Placement

In the conventional approach to the design of a single-input-single - output control system, we design a controller (compensator) such that the dominant closed-loop poles have desired damping ratio and an undammed natural frequency ω_n

In approach, the order of the system may be raised by 1 or 2 unless pole-zero cancellation takes place.

Note that in this approach we assume the effects on the responses of the non dominant closed-loop poles to be negligible.

Different from specifying dominant closed-loop poles (the convent - ional design approach), the present pole-placement approach specifies all closed-loop poles. There is also a requirement on the part of the system for the closed-loop poles to be placed any arbitrary chosen locations.

Consider a control system

$$X = Ax + Bu$$

$$Y = Cx + Du \qquad (1)$$

Where x = state vector (n-vector)

y = output signal (scalar)

u = control signal (scalar)

A = nxn constant matrix

B = nxl constant matrix

C = I xn constant matrix

D = constant (scalar), (Ogata, 2002).

We shall choose the control signal to be

$$\mathbf{u = -Kx}$$

This means that the control signal u is determined by an instantaneous state. Such a scheme is called state feedback.

The l x n matrix K is called the state feedback gain matrix. A block diagram for this system is shown in Figure (1).

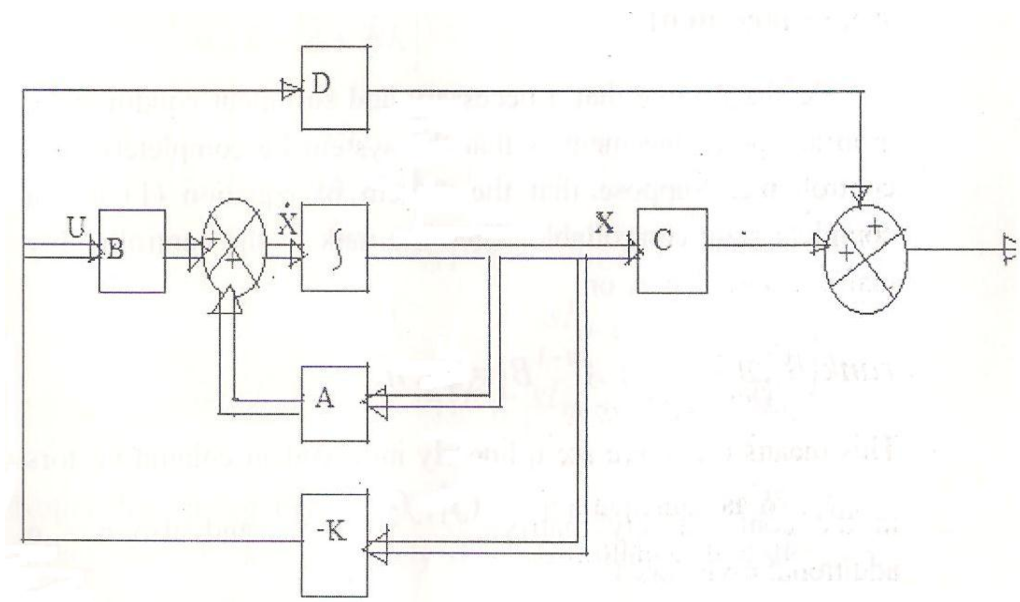

Figure (1): Closed-loop Control System with u = - kx.

Substituting equation (l) into equation (2) gives

X(t) = (A – BK) x(t) A which has the solution

$x(t) = e^{(A-KB)t}\, x(0)$ (3)

Where x(0) is the initial state caused by external disturbances.

The stability and transient response are determined by the eigenvalues of matrix A - BK (regulator poles). If these regulator poles are placed in the left-half of the s plane then x(t) approaches zero as approaches infinity. The problem of placing the regulator poles (closed-loop poles) at the desired location is called pole-placement problem (Ogata , 2002).

Necessary and Sufficient Condition for Arbitrary Pole Placement

We shall prove that a necessary and sufficient condition for Arbitrary pole-placement is that the system be completely state controllable – Suppose that the system of equation (l) is not complete state controllable. Then the rank of the controllability matrix is less than n, or

$$\text{rank } \lfloor B \vdots AB \vdots \ldots \vdots A^{n-1}B \rfloor = q < n$$

This means that there are q linearly independent column vectors in the controllability matrix $(f_1, f_2, \ldots f_q)$ and also n - q additional n-vectors

$v_{q+1}, v_{q+2,\ldots} v_n$ such that P= $|f_1 \vdots f_2 \vdots \cdots \vdots f_q \vdots v_{q+1} \vdots v_{q+2} \vdots \cdots \vdots v|_n$

is of rank n then it can be shown that

$$\grave{A} = P^{-1} AP = \begin{bmatrix} A_{11} \vdots A_{12} \\ 0 \vdots A_{22} \end{bmatrix} \quad \grave{B} = P^{-1}B = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}$$

Now we define $\grave{K} = KP = [K_1 \vdots K_2]$,then we have:

$$
\begin{aligned}
|sI - A + BK| &= \left|P^{-1}(sI - A + BK)P\right| \\
&= \left|sI - P^{-1}AP + P^{-1}BKP\right| \\
&= \left|sI - \hat{A} + \hat{B}\hat{K}\right| \\
&= \left|sI - \left[\begin{array}{c:c} A_{11} & A_{12} \\ 0 & A_{22} \end{array}\right] + \left[\begin{array}{c} B_{11} \\ 0 \end{array}\right]\left[K_1 \ \vdots \ K_2\right]\right| \\
&= \begin{vmatrix} sI_q - A_{11} + B_{11}K_1 & -A_{12} + B_{11}K_2 \\ 0 & sI_{n-q} - A_{22} \end{vmatrix} \\
&= \left|sI_q - A_{11} + B_{11}K_1\right| \cdot \left|sI_{n-q} - A_{22}\right| = 0
\end{aligned}
$$

Notice that the eigenvalues of 22 do not depends on Thus, If the system is not completely state controllable, then there are eigenvalues of matrix A that
cannot be arbitrary placed. Therefore, to place the eigenvalues of matrix A —BK arbitrary, the system must be completely state controllable (necessary condition).

Next we shall prove a sufficient condition (if the system is completely state controllable, then all eigenvalues of matrix A can be arbitrary placed. Define a transformation matrix T by

T=MW (4)

Where M is the controllability matrix

$$M = \lfloor B \vdots AB \vdots \cdots \vdots A^{n-1}B \rfloor \tag{5}$$

$$\text{And} \quad W = \begin{bmatrix} a_{n-1} a_{n-2} \ldots & a_1 & 0 \\ a_{n-2} a_{n-3} \ldots & 1 & 0 \\ \vdots \quad \vdots & \vdots & \vdots \\ a_1 \quad 1 \quad \ldots & 1 & 0 \\ 1 \quad 0 \quad \ldots & 0 & 0 \end{bmatrix} \tag{6}$$

where the a_i are coefficients of the characteristic polynomial

$$|sI - A| = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

Define a new state vector $\grave{x}$ by $x = T\grave{x}$ If . the rank of the controllability matrix M is n, it means that the system is completely state controllable, then T^{-1} exist and equation (l) can be written as

$$\dot{\hat{x}} = T^{-1}AT\,\hat{x} + T^{-1}Bu \tag{7}$$

Where

$$T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \tag{8}$$

$$T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (9)$$

Let us choose a set of the desired eigenvalues as $\mu_1, \mu_2, \ldots, \mu_n$, then the desired characteristic equation becomes

$$(s-\mu_1)(s-\mu_2)\cdots(s-\mu_n) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0 \quad (10)$$

Let us write:

$$KT = [\delta_1, \delta_2, \ldots, \delta_n] \quad (11)$$

When $u = KT\dot{x}$ is used to control the system given by equation (7), the system equation becomes $\dot{x} = T^{-1} AT\,\dot{x} - T^{-1}BKT\dot{x}$, the characteristic equation is $\left|sI - T^{-1}AT + T^{-1}BKT\right| = 0$,which is the same as the equation of the system.

Since $\dot{x} = Ax + Bu = (A - BK)x$, the characteristic equation for this system is

$$\left|sI - A + BK\right| = \left|T^{-1}(sI - A + BK)T\right| = \left|sI - T^{-1}AT + T^{-1}BKT\right| = 0$$

Now let us simplify the characteristic equation of the system in the controllable canonical, form. Referring to equation (8) and (9) we have

$$\left|sI - T^{-1}AT + T^{-1}BKT\right| = \left| sI - \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \delta_n & \delta_{n-1} & \cdots & \delta_1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} s & -1 & \cdots & 0 \\ 0 & s & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_n + \delta_n & a_{n-1} + \delta_{n-1} & \cdots & s + a_1 + \delta_1 \end{vmatrix}$$

$$= s^n + (a_1 + \delta_1)s^{n-1} + \cdots + (a_{n-1} + \delta_{n-1})s + (a_n + \delta_n) = 0 \qquad (12)$$

This is the characteristic equation for the system with state feedback. Therefore, it must be equal to equation (10), the desired characteristic equation. By equating the coefficients of the same power of s , we get,

$$\begin{aligned} a_1 + \delta_1 &= \alpha_1 \\ a_2 + \delta_2 &= \alpha_2 \\ &\vdots \\ a_n + \delta_n &= \alpha_n \end{aligned}$$

solving for δ_i and substituting in equation (l l) we obtain

$$K = \left[\delta_n \;\; \delta_{n-1} \;\; \cdots \;\delta_1\right]T^{-1}$$
$$= \left[\alpha_n - a_n \vdots \alpha_{n-1} - a_{n-1} \vdots \;\cdots\; \vdots \alpha_2 - a_2 \vdots \alpha_1 - a_1\right]T^{-1} \qquad (13)$$

Thus, if the system is completely state controllable, all Eigenvalues can be arbitrary placed by choosing matrix K according to equation (13) (sufficient condition). We have thus proved the necessary and sufficient condition for arbitrary pole placement is that the system be completely state controllable (Ogata, 2002).

Determination of matrix K using transformation matrix T

Suppose that the system is defined by $\dot{x}$ = Ax+Bu and the control signal u = -Kx. The feedback gain matrix K that forces the eigenvalues of A - BK to be μ_I , $\mu_2,\ldots\mu_n$ (desired values) can be determined by the following steps (if μ_I is a complex eigenvalue, then its conjugate must also be an eigenvalue):

Step 1: Check the controllability condition for the system. If the system is completely state controllable, then the following steps:

Step 2: From the characteristic polynomial for matrix A, that

$|sI - A| = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$ determine values of

a_1, a2,…,an

Step 3: Determine the transformation matrix T (that transforms the system state equation into the controllable canonical form (T = 1)

T is given by equation (4) or T=MW.

Step 4: Using the desired closed-loop poles to write the polynomial in equation (10) to find $a_1, a_2, \ldots, a_n$

Step 5: The required state feedback gain matrix K can be determined as in equation (13) (Ogata, 2002).

Determination of Matrix K Using Direct Substitution method

If the system is of low order n ≤ 3, direct substitution of matrix K into the desired characteristic polynomial may be simpler, for example if n = 3, then K = [kl k2 k3] substitute this K matrix into the desired characteristic polynomial $|sI - A + BK|$ and equate it to

$(s-\mu_1)(s-\mu_2)(s-\mu_3)$ or $|sI - A + BK|$ since both sides of this characteristic equation are polynomials in s by equating coefficients of the same power of s , it possible to determine the values of k_i. This approach is convenient if n = 2 or 3 (for n = 4 , 5, 6, … this approach may become very tedious). Note that if the system is not completely controllable, matrix K cannot be determined (No solution exist).

Determination of Matrix K Using Ackermann's formula

Ackermann's is a well-known formula, for determination of the state feedback gain matrix. To present this formula consider the system

$\dot{x}$ = Ax + Bu where u = - Kx. We assume that the system is completely State controllable, using the state feedback control (Sontag, 1998)

$$\dot{x} = (A - BK)x \quad (14)$$

Let us define $\tilde{A}$ = A - BK then the desired characteristic

$$|sI - ABK| = |sI - \tilde{A}| = (s - \mu_1)(s - \mu_2)\cdots(s - \mu_n)$$
$$= s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n = 0$$

where be μ_I , $\mu_2, \ldots \mu_n$ are the desired closed-loop poles. Since the Cayley – Hamilton theorem state that $\tilde{A}$ satisfies its own characteristic equation we have

$$\phi(\tilde{A}) = \tilde{A}^n + \alpha_1 \tilde{A}^{n-1} + \cdots + \alpha_{n-1}\tilde{A} + \alpha_n I = 0 \quad (15)$$

We shall utilize equation (15) to derive Ackermann's formula. To simplify

the derivation , we consider n = 3 and the following identities:

$$I = I$$
$$\widetilde{A} = A - BK$$
$$\widetilde{A}^2 = (A - BK)^2 - ABK - BK\widetilde{A}$$
$$\widetilde{A}^3 = (A - BK)^3 = A^3 - A^2BK - ABK\widetilde{A} - BK\widetilde{A}^2$$

multiplying the preceding equations by α_3, α_2, α_1 and $\alpha_0 = I$ respectively, and adding the results, we obtain:

$$\alpha_3 I + \alpha_2\widetilde{A} + \alpha_1\widetilde{A}^2 + \widetilde{A}^3$$
$$= \alpha_3 I + \alpha_2(A - BK) + \alpha_1\left(A^2 - ABK - BK\widetilde{A}\right) + A^3 - A^2BK - ABK\widetilde{A} - BK\widetilde{A}^2 \tag{16}$$

referring to equation (15), we have,

$$\alpha_3 I + \alpha_2\widetilde{A} + \alpha_1\widetilde{A}^2 + \widetilde{A}^3 = \phi(\widetilde{A}) = 0$$

Also we have, $$\alpha_3 I + \alpha_2 A + \alpha_1 A^2 + A^3 = \phi(A) \neq 0$$

substituting the last two equations into equation (16), we have

$$\phi(A) = \left[B \,\vdots\, AB \,\vdots\, A^2B\right]\begin{bmatrix} \alpha_2 K + \alpha_1 K\widetilde{A} + K\widetilde{A}^2 \\ \alpha_1 K + K\widetilde{A} \\ K \end{bmatrix} \tag{17}$$

Multiplying (17) by the inverse of the controllability matrix and then by

[0 0 1] we obtain

$$[0 \;\; 0 \;\; 1]\left[B \,\vdots\, AB \,\vdots\, A^2B\right]^{-1}\phi(A) = [0 \;\; 0 \;\; 1]\begin{bmatrix} \alpha_2 K + \alpha_1 K\widetilde{A} + K\widetilde{A}^2 \\ \alpha_1 K + K\widetilde{A} \\ K \end{bmatrix} = K$$

which can be written as

$$K = [0 \;\; 0 \;\; \cdots \;\; 0 \;\; 1]\left[B \,\vdots\, AB \,\vdots\, \cdots \,\vdots\, A^{n-1}B\right]^{-1}\phi(A) \tag{18}$$

and known as Ackermann s formula for the determination of the state feedback gain matrix K (Dingjum, 1997).

The Pole-Placement Problem

Consider the regulator system shown in Figure (2). Our chosen system is given by:

and the stste feedback control u = -Kx. Let us choose the

$$\dot{x} = Ax + Bu \text{ where } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and the stsre feedback control u = -Kx. Let us choose the desired closed-loop poles at

$$s = -2 + j4,\ s = -2 - j4,\ s = -10.$$

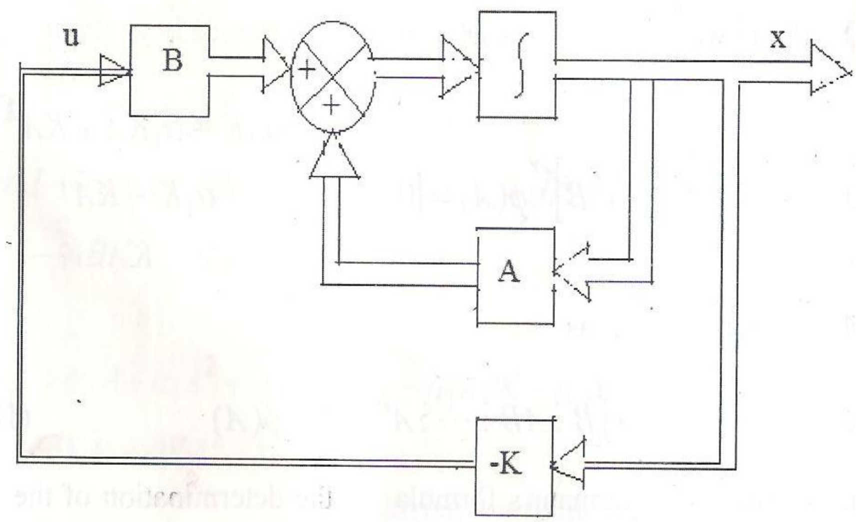

Figure (2): Regulator System

First, we need to check controllability matrix of the system M.

$$M = \left[B \vdots AB \vdots A^2B\right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$ we find

$|M|$ = -1 and rank M = 3. Thus, the system is state controllable And the arbitrary pole is possible. Next, we shall solve this problem. We shall demonstrate each of the four methods presented in this paper (Hamid, 2009).

Method 1: The first method is to use equation (13). The characteristic equation for the system is:

hence, $a_1 = 6$, $a_2 = 5$, $a_3 = I$ the desired characteristic equation is

$$(s+2-j4)(s+2+j4)(s+10) = s^3 + 14s^2 + 60s + 200$$

$$= s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = 0$$

hence, $\alpha_1 = 14$, $\alpha_2 = 60$, $\alpha_3 = 200$

Referring to equation (13), we have

$$K = [\alpha 3 - \alpha 3 \vdots \alpha 2 - \alpha 2 \vdots \alpha 1 - \alpha 1]^{T-1}$$ where T = I for this problem because the given state equation is in the controllable canonical form. Then, we have

$$K = [200-1 \vdots 60-5 \vdots 14-6]$$

$$= [\,199 \quad 55 \quad 8\,]$$

Method 2: By defining the desired state feedback gain matrix K **as**

k = $K_1 \quad K_2 \quad K_3$ and equating $|sI - A + BK|$ with the desired characteristic equation, we obtain:

$$|sI - A + BK| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [K_1 \quad K_2 \quad K_3] \right|$$

$$= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 + \mathrm{k}1 & 5 + k2 & s + 6 + k3 \end{vmatrix}$$

$$= s^3 + (6+k_3)s^2 + (5+k_2) + s + 1 + k_1$$

$$= s^3 + 14s^2 + 60s + 200$$

Hence, $6+k_3 = 14$, $5+k_2 = 60$, $I + k_l = 200$ from which we obtain

$k_1=199$, $K2 = 55$, $K3 = 8$ or $K = [199 \quad 55 \quad 8]$

Method 3: The third method is to use Ackermann s formula. Referring to equation (18), we have:

$$K = [0 \quad 0 \quad 1][B \vdots AB \ \vdots A^2B]^{-1} \phi\, A$$

since ϕ (A) = A3 + $14A^2$ + 60 A + 200

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0 & -5 & -6 \end{bmatrix}^3 + 14\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^2 + 60\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}$$

$$+ 200\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} = \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

And

$$[B\colon \quad AB : \quad A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix} \text{ and we obtain:}$$

$$K = [0 \quad 0 \quad 1]\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}^{1-} \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$= [0 \quad 0 \quad 1]\begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$= [199 \quad 55 \quad 8]$$

Solving Pole-Placement Problem with MAT LAB

MATLAB has two commands - acker and place – for the computation of the feedback-gain matrix K. The command acker is based on Ackerman's formula. This command applies to single-input systems only. The pole-placement based on the approach of maximize stability margin is called the robust pole placement and the MTLAB command for it is called place (Ogata, 2002).

To use command 'acker or place, we first enter the following matrices in the program:

A matrix , B matrix , J matrix

where J matrix is consisting of the desired closed-loop poles such that

$$J = [\ \mu_1\ ,\ \mu_2, \ldots, \mu_n],$$

Then we enter K = acker (A,B,J) or K = place (A,B,J). It is noted that the command eig(A-B*K) used to verify that K obtain gives the desired eigenvalues.

We considered the same system:

$$X = Ax + Bu \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

By using state feedback control $u = -Kx$, it is desired to have the closed- loop poles.

$s = \mu_1$, $s = \mu_2$, $s = \mu_3$ where $\mu_1 = -2 + j4$, $\mu 2 - 2 - j4$, $\mu 3 = -10$

Determine the state feedback-gain matrix K with MATLAB.

MAT LAB programs that generate matrix K are shown in MATLAB (1) and (2) MATLAB program (1) uses command acker and program (2) uses command place (Ogata, 2002).

MATLAB Program (l)
A = [0 1 0:0 0 1 ; -1 -5 -6];
B = [0;0;1];
J = [-2+j*4 -2 –j*4 -10];
K =acker(A,B,J)
K =
199 55 8

MATLAB Program (2)
A = [0 1 0:0 0 1 ; -1 -5 -6];
B = [0;0;1];
J = [-2+j*4 -2 –j*4 -10];
K = place(A,B,J)
Place: ndigits = 15
K =
199 .0000 55.00000 8.0000

Conclusion

As a matter of course, the feedback gain matrix K obtained by the four methods are the same. With this state feedback, the closed-loop poles are placed at s = -2 ± j4 and s-10, as desired.

It is noted that if the order n of the system were 4 or higher, methods I and 3 are recommended, since all matrix computations can be carried out by a computer. If method 2 is used, hand computations become necessary because a computer may not handle a characteristic equation with unknown parameters $K_1,K_2,\ldots,K_n$.

If the system is of the second order, then the system dynamics (response characteristics) can be precisely correlated to the location of the desired closed-loop poles and the zero(s) of the considered system.

In determining the state feedback gain matrix K for a given system, it is desirable to examine by computer simulations the response characteristic of the system for different several matrices K and to choose the one that gives the best overall system performance.

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